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Homework 4

Due February 8th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake. There are some hints on the second page.

1. Formulate and prove analogues of Borthwick Theorem 4.12 and Corollary 4.13 for the Klein–Gordon equation (see the end of the chapter). What still works and what goes wrong if the m^2u term is replaced by $-m^2u$?
2. Let $u \in C^2([0,\infty) \times \mathbb{R}^3)$ solve $\partial_t^2 u = \Delta u$ with $u(0,x) = \partial_t u(0,x) = 0$ when $|x| \geq 2$. For which nonnegative values of t and $|x|$ is u guaranteed to be 0? Sketch the region in the first quadrant of the $(|x|,t)$ plane.
3. How does the answer to the above problem change if $\mathbb{R}^3$ is replaced by $\mathbb{R}^2$ or $\mathbb{R}$? Redo the sketch for these cases.
4. Let g and h be in $C^2(\mathbb{R})$.

(Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose g, h have compact support. The *kinetic energy* is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and the *potential energy* is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$. Prove

(a) $k(t) + p(t)$ is constant in t,

(b) $k(t) = p(t)$ for all large enough times t.

Hints:

1. Add a term to the energy $\mathcal{E}[u](t)$ in such a way that $\frac{d}{dt}\mathcal{E}[u](t) = 0$. The needed changes to the proofs in the book are minor.

2. and 3. Use Huygens principle and/or the integral solution formulas for u.

4. For (a), follow the proof of Borthwick Theorem 4.12. For (b), plug the form (4.14) into $k(t)-p(t)$ and simplify. This exercise comes from Evans' book.